4. Metoda parcijalne integracije

Prema formuli diferenciranja znamo da je

$$d(uv) = u\,dv + v\,du$$

a znamo i da je $\int d(uv) = uv$. Prema tome nije teško vidjeti da vrijedi

$$\int u\,dv = uv - \int v\,du \qquad \ldots (*)$$

Formulu $(*)$ nazivamo formula parcijalne integracije.

Posmatrajmo neki dati integral $\int f(x)\,g(x)\,dx$. Metodom parcijalne integracije biramo f-ju $u = f(x)$ i $dv = g(x)\,dx$ tako da se integral $\int v\,du$ može jednostavno riješiti.

(#) Odrediti integrale

a) $\int x\cos x \, dx$ b) $\int \frac{\ln x}{x^3} dx$ c) $\int x \operatorname{arctg} x \, dx$

d) $\int \arcsin x \, dx$ e) $\int x^2 e^{3x} dx$ f) $\int e^{-x} \cos\frac{x}{2} dx$

Rj.

a) $$\int x\cos x \, dx = \left| \begin{array}{ll} u = x & dv = \cos x \, dx \\ du = dx & v = \int \cos x \, dx = \sin x \end{array} \right| =$$

$$= x\sin x - \int \sin x \, dx = x\sin x + \cos x + C$$

b) $$\int \frac{\ln x}{x^3} dx = \left| \begin{array}{ll} u = \ln x & dv = \frac{dx}{x^3} \\ du = \frac{1}{x} dx & v = \int \frac{dx}{x^3} = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} \end{array} \right| =$$

$$= -\frac{\ln x}{2x^2} - \int \frac{(-1)}{2x^2} \cdot \frac{1}{x} dx = -\frac{\ln x}{2x^2} + \frac{1}{2} \int \frac{dx}{x^3} =$$

$$= -\frac{1}{2x^2} \ln x + \frac{1}{2} \cdot \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C$$

$$= C - \frac{2\ln x + 1}{4x^2} + C$$

c) $$\int x \operatorname{arctg} x \, dx = \left| \begin{array}{ll} u = \operatorname{arctg} x & dv = x \, dx \\ du = \frac{dx}{1+x^2} & v = \int x \, dx = \frac{x^2}{2} \end{array} \right| =$$

$$= \frac{1}{2} x^2 \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$ Izračunajmo posebno $\int \frac{x^2}{1+x^2} dx$

... (1)

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx =$$

$$= \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \operatorname{arctg} x + c_1$$

Sad, prema (1) imamo

$$I = \frac{1}{2} x^2 \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x + c =$$

$$= c - \frac{1}{2} x + \frac{x^2+1}{2} \operatorname{arctg} x$$

d) $$\int \arcsin x\,dx = \begin{vmatrix} u = \arcsin x & dv = dx \\ du = \dfrac{dx}{\sqrt{1-x^2}} & v = \int dx = x \end{vmatrix} =$$

$$= x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = \begin{vmatrix} d(1-x^2) = -2x\,dx \\ x\,dx = -\frac{1}{2} d(1-x^2) \end{vmatrix} =$$

$$= x \arcsin x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2)$$

$$= x \arcsin x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \arcsin x + \sqrt{1-x^2} + c$$

e) $$\int x^2 e^{3x}\,dx = \begin{vmatrix} u = x^2 & dv = e^{3x}\,dx \\ du = 2x\,dx & v = \int e^{3x}\,dx = \frac{1}{3}\int e^{3x}\,d(3x) = \frac{1}{3} e^{3x} \end{vmatrix} =$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x\, e^{3x}\,dx \qquad \dots (1)$$

$$\int x e^{3x}\,dx = \begin{vmatrix} u = x & dv = e^{3x}\,dx \\ du = dx & v = \frac{1}{3} e^{3x} \end{vmatrix} = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x}\,dx =$$

$$= \frac{1}{3} x e^{3x} - \frac{1}{3} \cdot \frac{1}{3} e^{3x} + C_1 = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + C_1$$

Prema (1) imamo

$$\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{2x} + C_1\right) =$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C$$

$$= \frac{1}{27} e^{3x} \left(9x^2 - 6x + 2\right) + C$$

f) $$I = \int e^{-x} \cos\frac{x}{2}\, dx = \left| \begin{array}{ll} u = e^{-x} & dv = \cos\frac{x}{2}\, dx \\ du = e^{-x} \cdot (-1) = -e^{-x} dx & v = \int \cos\frac{x}{2}\, dx = 2\int \cos\frac{x}{2}\, d\left(\frac{x}{2}\right) \\ & \quad = 2 \sin\frac{x}{2} \end{array} \right|$$

$$= 2e^{-x} \sin\frac{x}{2} + 2\int e^{-x} \sin\frac{x}{2}\, dx = \left| \begin{array}{ll} u = e^{-x} & dv = \sin\frac{x}{2}\, dx \\ du = -e^{-x} dx & v = (-2) \cos\frac{x}{2} \end{array} \right| =$$

$$= 2e^{-x} \sin\frac{x}{2} + 2\left((-2)\, e^{-x} \cos\frac{x}{2} - 2\int e^{-x} \quad s\frac{x}{2}\, dx\right)$$

$$= 2e^{-x} \sin\frac{x}{2} - 4e^{-x} \cos\frac{x}{2} - 4\int e^{-x} \cos\frac{x}{2}\, dx$$

Tj.

$$I = 2e^{-x} \sin\frac{x}{2} - 4e^{-x} \cos\frac{x}{2} - 4I$$

$$5I = 2e^{-x} \sin\frac{x}{2} - 4e^{-x} \cos\frac{x}{2}$$

$$I = \frac{2}{5} e^{-x} \left(\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) + C$$

Zadaci za vježbu

1. $\int x\sin x\,dx$ 2. $\int x^2\ln x\,dx$ 3. $\int \ln(x^n)\,dx$

4. $\int (x^2+1)e^{-2x}\,dx$ 5. $\int \frac{x}{\cos^2 x}\,dx$ 6. $\int x\ln(x-1)\,dx$

7. $\int \operatorname{arc\,ctg} t\,dt$ 8. $\int \ln(1+x^2)\,dx$ 9. $\int e^{ax}\sin bx\,dx$

10.* $\int \frac{\arcsin x}{x^2}\,dx$ 11.* $\int \frac{\ln x\,dx}{(x+1)^2}$

12.* $\int \operatorname{arc\,tg}\sqrt{2x-1}\,dx$

Rješenja

1. $\sin x - x\cos x$ 2. $\frac{x^3}{9}(3\ln x-1)$ 3. $nx(\ln x-1)$

4. $-\frac{2x^2+2x+3}{4e^{2x}}$ 5. $x\operatorname{tg}x+\ln|\cos x|$ 6. $\frac{x^2-1}{2}\ln|x-1|-$
$-\frac{x^2}{4}-\frac{x}{2}$ 7. $t\operatorname{arc\,ctg}t+\frac{1}{2}\ln(1+t^2)$

8. $x\ln(x^2+1)-2x+2\operatorname{arc\,tg}x$ 9. $\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

10. $\ln\frac{1-\sqrt{1-x^2}}{x}-\frac{1}{x}\arcsin x$ 11. $\frac{x\ln x}{x+1}-\ln|x+1|$

12. $x\operatorname{arc\,tg}\sqrt{2x-1}-\frac{1}{2}\sqrt{2x-1}$

Izabrani **Zadaci za vježbu** sa rješenjima (iz lekcije Metoda parcijalne integracije)

1.
$$\int x e^x\,dx = \begin{vmatrix} u=x & dv=e^x dx \\ du=dx & v=\int e^x dx = e^x \end{vmatrix} = xe^x - \int e^x dx = xe^x - e^x + C$$

Da smo uzeli smjene $u=e^x$, $dv = x\,dx$ dobili bi komplikovan izraz.

2.
$$\int x^2 \sin 3x\,dx = \begin{vmatrix} u=x^2 & dv=\sin 3x\,dx \\ du=2x\,dx & v=\int \sin 3x\,dx = -\frac{1}{3}\cos 3x \end{vmatrix} \overset{(*)}{=}$$

$$\int \sin 3x\,dx = \begin{vmatrix} 3x=t \\ 3\,dx=dt \\ dx=\frac{1}{3}dt \end{vmatrix} = \frac{1}{3}\int \sin t\,dt = -\frac{1}{3}\cos t + c = -\frac{1}{3}\cos 3x + c$$

$$\overset{(*)}{=} -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3} I_1$$

$$I_1 = \int x\cos 3x\,dx = \begin{vmatrix} u=x & dv=\cos 3x\,dx \\ du=dx & v=\frac{1}{3}\sin 3x \end{vmatrix} = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\cdot\left(-\frac{1}{3}\right)\cos 3x + C_1 = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C_1$$

$$\int x^2\sin 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C$$

3.
$$\int x^3 \ln x\,dx = \begin{vmatrix} u=\ln x & dv=x^3 dx \\ du=\frac{1}{x}dx & v=\frac{x^4}{4} \end{vmatrix} = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + c = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

4.
$$\int \arcsin x\,dx = \begin{vmatrix} u=\arcsin x & dv=dx \\ du=\frac{1}{\sqrt{1-x^2}}dx & v=x \end{vmatrix} = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\arcsin x - I_1$$

$$I_1=\int \frac{x}{\sqrt{1-x^2}}\,dx=\left|\begin{array}{c} 1-x^2=t^2 \\ -2x\,dx=2t\,dt \\ x\,dx=-t\,dt \\ t=\sqrt{1-x^2}\end{array}\right|=\int\frac{-t\,dt}{t}=-\int dt=-t+C_1=$$
$$=-\sqrt{1-x^2}+C_1$$

$$\int \arcsin x\,dx = x\arcsin x+\sqrt{1-x^2}+C$$

5. ISPITNI ZADATAK

$$\int \sin(\ln x)\,dx=\left|\begin{array}{ll} u=\sin(\ln x) & dv=dx \\ du=\cos(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{array}\right|=x\sin(\ln x)-\int\cos(\ln x)\,dx$$

$$\int \cos(\ln x)\,dx=\left|\begin{array}{ll} u=\cos(\ln x) & dv=dx \\ du=-\sin(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{array}\right|=x\cos(\ln x)+\int\sin(\ln x)\,dx$$

$$\underline{\underline{\int \sin(\ln x)\,dx}}=x\sin(\ln x)-x\cos(\ln x)-\underline{\underline{\int\sin(\ln(x))\,dx}}$$

$$2\int\sin(\ln x)\,dx=x\sin(\ln x)-x\cos(\ln x)\quad /:2$$

$$\int\sin(\ln x)\,dx=\frac{x}{2}\sin(\ln x)-\frac{x}{2}\cos(\ln x)+C$$

6. $$\int x\,\mathrm{arctg}\,x\,dx=\left|\begin{array}{ll} u=\mathrm{arctg}\,x & dv=x\,dx \\ du=\frac{1}{1+x^2}\,dx & v=\frac{x^2}{2}\end{array}\right|=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$\int\frac{x^2}{x^2+1}\,dx=\int\frac{x^2+1-1}{x^2+1}\,dx=\int dx-\int\frac{1}{x^2+1}\,dx=x-\mathrm{arctg}\,x+C_1$$

$$\int x\,\mathrm{arctg}\,x\,dx=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}x+\frac{1}{2}\mathrm{arctg}\,x+C$$

7. V $\int x^2e^{-2x}\,dx$ Rj: $-\frac{x^2}{2}e^{-2x}-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C$

8. V $\int\frac{x^2}{(x^2+1)^2}\,dx$ Rj: $-\frac{x}{2(x^2+1)}+\frac{1}{2}\mathrm{arctg}\,x+C$

uputa: $\int x\cdot\frac{x}{(x^2+1)^2}\,dx=\left|\begin{array}{ll} u=x & dv=\frac{x\,dx}{(x^2+1)^2}\end{array}\right|=\ldots$

9.

$$\int x^3 \ln(2x+1)\,dx = \left| \begin{matrix} u=\ln(2x+1) & dv = x^3\,dx \\ du = \frac{1}{2x+1}\cdot 2\,dx & v=\frac{x^4}{4} \end{matrix} \right| =$$

$$= \frac{1}{4}x^4\ln(2x+1) - \frac{1}{4}\cdot 2\int \frac{x^4}{2x+1}\,dx = \frac{1}{4}x^4\ln(2x+1) - \frac{1}{2}I_1$$

$$x^4 : (2x+1) = \frac{1}{2}x^3 - \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{16}$$

$$\begin{array}{l} -\ x^4 + \frac{1}{2}x^3 \\ \hline -\frac{1}{2}x^3 \\ -\ -\frac{1}{2}x^3 - \frac{1}{4}x^2 \\ \hline \frac{1}{4}x^2 \\ -\ \frac{1}{4}x^2 + \frac{1}{8}x \\ \hline -\frac{1}{8}x \\ -\ -\frac{1}{8}x - \frac{1}{16} \\ \hline \text{ostatak } \frac{1}{16} \end{array}$$

ostatak $\frac{1}{16}$

$$x^4 = \left(\frac{1}{2}x^3 - \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{16}\right)(2x+1) + \frac{1}{16}$$

$$I_1 = \int \left(\frac{1}{2}x^3 - \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{16} + \frac{\frac{1}{16}}{2x+1}\right)dx =$$

$$= \frac{1}{2}\cdot\frac{x^4}{4} - \frac{1}{4}\cdot\frac{x^3}{3} + \frac{1}{8}\cdot\frac{x^2}{2} - \frac{1}{16}x + \frac{1}{16}\cdot\frac{1}{2}\ln|2x+1| + C_1$$

$$\int x^3\ln(2x+1)\,dx = \frac{1}{4}x^4\ln(2x+1) - \frac{1}{16}x^4 + \frac{1}{24}x^3 - \frac{1}{32}x^2 + \frac{1}{32}x - \frac{1}{64}\ln|2x+1| + C$$

10. ISPITNI ZADATAK

$$I = \int \sqrt{x}\,\ln^2 x\,dx = \left| \begin{matrix} u=\ln^2 x & dv=\sqrt{x}\,dx \\ du = 2\ln x\cdot\frac{1}{x}\,dx & v=\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{x^3} \end{matrix} \right| =$$

$$= \frac{2}{3}\sqrt{x^3}\ln^2 x - 2\cdot\frac{2}{3}\int \frac{\sqrt{x^3}}{x}\ln x\,dx = \frac{2}{3}x\sqrt{x}\ln^2 x - \frac{4}{3}\int\sqrt{x}\ln x\,dx$$

$$\int \sqrt{x}\ln x\,dx = \left| \begin{matrix} u=\ln x & dv=\sqrt{x}\,dx \\ du=\frac{1}{x}\,dx & v=\frac{2}{3}\sqrt{x^3} \end{matrix} \right| = \frac{2}{3}x\sqrt{x}\ln x - \frac{2}{3}\int\frac{x\sqrt{x}}{x}\,dx =$$

$$= \frac{2}{3}x\sqrt{x}\ln x - \frac{2}{3}\cdot\frac{2}{3}\sqrt{x^3} + C = \frac{2}{3}x\sqrt{x}\ln x - \frac{4}{9}x\sqrt{x} + C$$

$$I = \frac{2}{3}x\sqrt{x}\ln^2 x - \frac{8}{9}x\sqrt{x}\ln x + \frac{16}{27}x\sqrt{x} + C$$

11. ISPITNI ZADATAK

$$I=\int e^{3x}\cos 4x\,dx=\begin{vmatrix} u=e^{3x} & dv=\cos 4x\,dx \\ du=3e^{3x}dx & v=\frac{1}{4}\sin 4x \end{vmatrix}=$$

$$=\frac{1}{4}e^{3x}\sin 4x-\frac{3}{4}\int e^{3x}\sin 4x\,dx=\frac{1}{4}e^{3x}\sin 4x-\frac{3}{4}I_1$$

$$I_1=\int e^{3x}\sin 4x\,dx=\begin{vmatrix} u=e^{3x} & dv=\sin 4x\,dx \\ du=3e^{3x}dx & v=-\frac{1}{4}\cos 4x \end{vmatrix}=$$

$$=-\frac{1}{4}e^{3x}\cos 4x+\frac{3}{4}\int e^{3x}\cos 4x\,dx=-\frac{1}{4}e^{3x}\cos 4x+\frac{3}{4}I$$

$$I=\frac{1}{4}e^{2x}\sin 4x+\frac{3}{16}e^{3x}\cos 4x-\frac{9}{16}I \qquad /\cdot 16$$

$$16I=4e^{2x}\sin 4x+3e^{3x}\cos 4x-9I$$

$$I=\frac{4e^{3x}\sin 4x+3e^{3x}\cos 4x}{25}+C$$

12. ISPITNI ZADATAK

$$I=\int x^2e^{3x}dx=\begin{vmatrix} u=x^2 & dv=e^{3x}dx \\ du=2x\,dx & v=\frac{1}{3}e^{3x} \end{vmatrix}=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\int xe^{3x}dx$$

$$\int xe^{3x}dx=\begin{vmatrix} u=x & dv=e^{3x}dx \\ du=dx & v=\frac{1}{3}e^{3x} \end{vmatrix}=\frac{1}{3}xe^{3x}-\frac{1}{3}\int e^{3x}dx=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$$

$$I=\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+C$$

13. $I=\int\frac{\arcsin\frac{x}{2}}{\sqrt{2-x}}dx$ Uputa: $u=\arcsin\frac{x}{2}$, $dv=\frac{dx}{\sqrt{2-x}}$ Rj. $-2\sqrt{2-x}\cdot\arcsin\frac{x}{2}+4\sqrt{2+x}+C$

14. $I=\int e^x\sin x\,dx$ Uputa: $u=e^x$ Rj. $\frac{e^x(\sin x-\cos x)}{2}+C$

15. $\int x^5\ln x\,dx$

16. $\int\frac{x^2\,dx}{\cos^2 x}$

17. $\int\frac{\ln(x^2+1)}{x^3}dx$

18. $\int(\arcsin x)^2dx$

(#) Izračunati integral $I=\int x^3 e^{\frac{x}{2}}\,dx$.

Rj.

$$\int e^{\frac{x}{2}}dx = \left| \begin{array}{l} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2\,dt \end{array} \right| = 2\int e^t dt = 2e^t + c = 2e^{\frac{x}{2}} + c$$

$$\int x^3 e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x^3 & dv=e^{\frac{x}{2}}dx \\ du=3x^2dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2x^3e^{\frac{x}{2}} - 6\int x^2 e^{\frac{x}{2}}dx$$

$$\int x^2 e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x^2 & dv=e^{\frac{x}{2}}dx \\ du=2x\,dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2x^2e^{\frac{x}{2}} - 4\int x e^{\frac{x}{2}}dx$$

$$\int x e^{\frac{x}{2}}dx = \left| \begin{array}{ll} u=x & dv=e^{\frac{x}{2}}dx \\ du=dx & v=2e^{\frac{x}{2}} \end{array} \right| = 2xe^{\frac{x}{2}} - 2\int e^{\frac{x}{2}}dx = 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c$$

$$I = 2x^3e^{\frac{x}{2}} - 6\left[2x^2e^{\frac{x}{2}} - 4\left(2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}}\right)\right] + c =$$

$$= 2x^3e^{\frac{x}{2}} - 6\left(2x^2e^{\frac{x}{2}} - 8xe^{\frac{x}{2}} + 16e^{\frac{x}{2}}\right) + c$$

$$= 2x^3e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$$

$$I = 2e^{\frac{x}{2}}\left(x^3 - 6x^2 + 24x - 48\right) + c$$

(#) Izračunati: $\int \sqrt{5-x^2}\,dx$.

Rj. $$\int \sqrt{5-x^2}\,dx = \left| \begin{array}{l} x=\sqrt{5}\,t \\ dx=\sqrt{5}\,dt \\ t=x/\sqrt{5} \end{array} \right| = \int \sqrt{5-5t^2}\,\sqrt{5}\,dt = 5\int\sqrt{1-t^2}\,dt$$

$$\int\sqrt{1-t^2}\,dt = \left| \begin{array}{ll} u=\sqrt{1-t^2} & dv=dt \\ du=\frac{-2t}{2\sqrt{1-t^2}}=\frac{-t}{\sqrt{1-t^2}}dt & v=t \end{array} \right| = t\sqrt{1-t^2}+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$\int\frac{t^2}{\sqrt{1-t^2}} = \int\frac{t^2-1+1}{\sqrt{1-t^2}}dt = -\int\frac{1-t^2}{\sqrt{1-t^2}}dt+\int\frac{dt}{\sqrt{1-t^2}} =$$

$$= -\int\sqrt{1-t^2}\,dt + \arcsin t$$

$$\int\sqrt{1-t^2}\,dt = t\sqrt{1-t^2} - \int\sqrt{1-t^2} + \arcsin t$$

$$2\int\sqrt{1-t^2}\,dt = t\sqrt{1-t^2} + \arcsin t$$

$$\int\sqrt{1-t^2} = \frac{1}{2}\left(t\sqrt{1-t^2}+\arcsin t\right) + C$$

$$\int\sqrt{5-x^2}\,dx = \frac{5}{2}\left(\frac{x}{\sqrt{5}}\sqrt{1-\frac{x^2}{5}}+\arcsin\frac{x}{\sqrt{5}}\right)+C =$$

$$= \frac{5}{2}\left(\frac{x\sqrt{5}}{5}\sqrt{\frac{5-x^2}{5}}+\arcsin\frac{x\sqrt{5}}{5}\right)+C =$$

$$= \frac{x}{2}\cdot\sqrt{5-x^2} + \frac{5}{2}\arcsin\frac{x\sqrt{5}}{5} + C$$

(#) Izračunati integral $\int x\sqrt{1-x^4}\,dx$

Rj.

$$\int x\sqrt{1-x^4}\,dx = \int x\sqrt{(1-x^2)(1+x^2)}\,dx = \left| \begin{array}{l} x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{array} \right| = \frac{1}{2}\int\sqrt{(1-t)(1+t)}\,dt =$$

$$= \frac{1}{2}\int\sqrt{1-t^2}\,dt = \frac{1}{2}\int\frac{1-t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\left[\int\frac{dt}{\sqrt{1-t^2}} - \int\frac{t^2}{\sqrt{1-t^2}}\,dt\right]$$

$$\int\frac{t^2}{\sqrt{1-t^2}}\,dt = \left| \begin{array}{ll} u=t & dv=\frac{t}{\sqrt{1-t^2}}dt \\ du=dt & v=\int\frac{t}{\sqrt{1-t^2}}dt = \left| \begin{array}{l} 1-t^2=s^2 \\ -2t\,dt=2s\,ds \\ t\,dt=-s\,ds \end{array} \right| = -\int\frac{s\,ds}{s} = -\int ds \\ & = -s = -\sqrt{1-t^2} \end{array} \right| =$$

$$= -t\sqrt{1-t^2} + \int\sqrt{1-t^2}\,dt$$

Sad imamo:

$$\underline{\underline{\frac{1}{2}\int\sqrt{1-t^2}\,dt}} = \frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2} - \underline{\underline{\frac{1}{2}\int\sqrt{1-t^2}\,dt}}$$

$$\int\sqrt{1-t^2}\,dt = \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t$$

$$\frac{1}{2}\int\sqrt{1-t^2} = \frac{1}{2}\left(\frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t\right)$$

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$$\int x\sqrt{1-x^4}\,dx = \frac{1}{4}x^2\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + C$$